Automatic continuity notions and locally compact Polish groups

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When Topological Dynamics meets Model Theory

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- If the algebraic structure completely determines the topological structure, then any such ψ must be continuous.
- By placing additional conditions on H, we can qualify to what extent the topology is determined.

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Remark

None of these examples are locally compact.

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Corollary

Every non-trivial connected locally compact Polish group fails the SIP.

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The **n-th level rigid stabilizer** of G is $\mathrm{st}_G(n) := \langle \mathrm{rist}_G(s) \mid |s| = n\rangle$.

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Example

The inverse limit $W(\mathrm{Alt}(5)) := \varprojlim ((\mathrm{Alt}(5), [4]) \wr \cdots \wr (\mathrm{Alt}(5), [4]))$ is a sji profinite branch group.

Automatic continuity results

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Fact (Borel; Kallman)

$PSL_3(\mathbb{Z}_p)$ is a strongly just infinite profinite group but fails the SIP.

A sequence of symmetric subsets $(B_i)_{i\in\mathbb{N}}$ in a Polish group G is called a **Bergman sequence** if $G = \bigcup_{i\in\mathbb{N}} B_i$ and $B_iB_i \subseteq B_{i+1}$ for all i.

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If G is a strongly just infinite profinite branch group, then G admits exactly two locally compact group topologies:

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If G is a strongly just infinite profinite branch group, then G admits exactly two locally compact group topologies: the discrete topology and the usual topology.

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There is a non-discrete topologically simple locally compact Polish group with the SIP, the invariant automatic continuity property, and the locally compact automatic continuity property. E.g. the Burger-Mozes universal groups $U(A_n)^+$ for $n \geq 6$.

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- Suppose $(F_i, X_i)_{i\in\mathbb{N}}$ is a sequence of non-trivial finite perfect permutation groups so that F_i acts transitively and faithfully on X_i. Suppose further there is $C \geq 1$ so that $\mathrm{cw}(F_i) \leq C$ for all i. (Any sequence of non-abelian finite simple groups suffices.)
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Remarks and Questions

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Question

If G is a strongly just infinite profinite Polish group with the Bergman property, then does G have the SIP?

Thank you